73. *Telephone rates.* Refer to Problems 71 and 72. Write a brief verbal comparison of the two services described for calls lasting 20 minutes or less.

74. *Telephone rates.* Refer to Problems 71 and 72. Write a brief verbal comparison of the two services described for calls lasting more than 20 minutes.

A company sells custom embroidered apparel and promotional products. The table that follows shows the volume discounts offered by the company, where x is the volume of a purchase in dollars. Problems 75 and 76 deal with two different interpretations of this discount method.

Volume Discount (Excluding Tax)	
Volume ($x)	**Discount Amount**
$\$300 \le x < \$1{,}000$	3%
$\$1{,}000 \le x < \$3{,}000$	5%
$\$3{,}000 \le x < \$5{,}000$	7%
$\$5{,}000 \le x$	10%

75. *Volume discount.* Assume that the volume discounts in the table apply to the entire purchase. That is, if the volume x satisfies $\$300 \le x < \$1{,}000$, the entire purchase is discounted 3%. If the volume x satisfies $\$1{,}000 \le x < \$3{,}000$, the entire purchase is discounted 5%, and so on.

(A) If x is the volume of a purchase before the discount is applied, write a piecewise definition for the discounted price $D(x)$ of this purchase.

(B) Use one-sided limits to investigate the limit of $D(x)$ as x approaches $1,000; as x approaches $3,000.

76. *Volume discount.* Assume that the volume discounts in the table apply only to that portion of the volume in each interval in the table. That is, the discounted price for a $4,000 purchase would be computed as follows:

$$300 + 0.97(700) + 0.95(2{,}000) + 0.93(1{,}000) = 3{,}809$$

(A) If x is the volume of a purchase before the discount is applied, write a piecewise definition for the discounted price $P(x)$ of this purchase.

(B) Use one-sided limits to investigate the limit of $P(x)$ as x approaches $1,000; as x approaches $3,000.

(C) Compare this discount method with the one in Problem 75. Does one always produce a lower price than the other? Discuss.

77. *Pollution.* A state charges polluters an annual fee of $20 per ton for each ton of pollutant emitted into the atmosphere, up to a maximum of 4,000 tons. No fees are charged for emissions beyond the 4,000-ton limit. Write a piecewise definition of the fees $F(x)$ charged for the emission of x tons of pollutant in a year. What is the limit of $F(x)$ as x approaches 4,000 tons? As x approaches 8,000 tons?

78. *Pollution.* Refer to Problem 77. The fee per ton of pollution is given by $A(x) = F(x)/x$. Write a piecewise definition of $A(x)$. What is the limit of $A(x)$ as x approaches 4,000 tons? As x approaches 8,000 tons?

79. *Voter turnout.* Statisticians often use piecewise-defined functions to predict outcomes of elections. For the following functions f and g, find the limit of each function as x approaches 5 and as x approaches 10:

$$f(x) = \begin{cases} 0 & \text{if } x \le 5 \\ 0.8 - 0.08x & \text{if } 5 < x < 10 \\ 0 & \text{if } 10 \le x \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if } x \le 5 \\ 0.8x - 0.04x^2 - 3 & \text{if } 5 < x < 10 \\ 1 & \text{if } 10 \le x \end{cases}$$

Section 10-2 CONTINUITY

- Continuity
- Continuity Properties
- Solving Inequalities by Using Continuity Properties

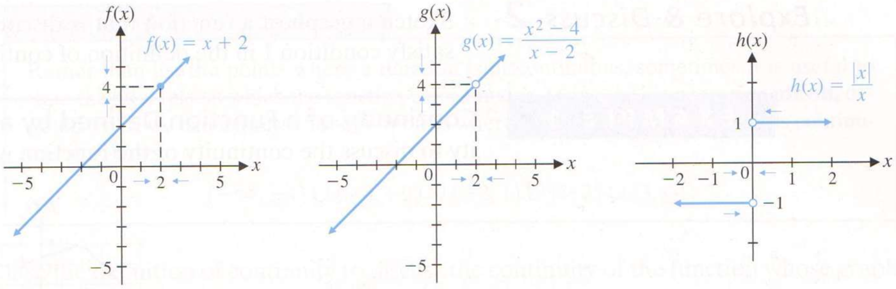
Theorem 3 in Section 10-1 states that if f is a polynomial function or a rational function with a nonzero denominator at $x = c$, then

$$\lim_{x \to c} f(x) = f(c) \tag{1}$$

Functions that satisfy equation (1) are said to be *continuous* at $x = c$. A firm understanding of continuous functions is essential for sketching and analyzing graphs. We will also see that continuity properties provide a simple and efficient method for solving inequalities—a tool that we will use extensively in later sections.

Continuity

Compare the graphs shown in Figure 1. Notice that two of the graphs are broken; that is, they cannot be drawn without lifting a pen off the paper. Informally, a function is *continuous over an interval* if its graph over the interval can be drawn without removing a pen from the paper. A function whose graph is broken (disconnected) at



(A) $\lim_{x\to 2} f(x) = 4$
$f(2) = 4$

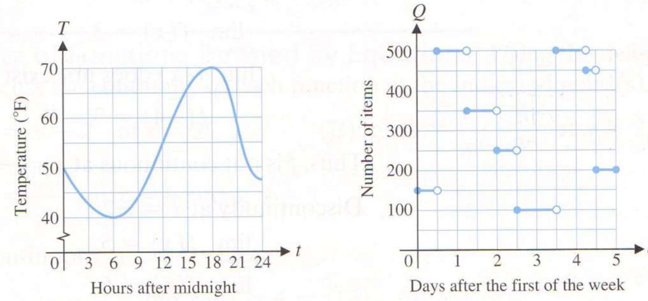
(B) $\lim_{x\to 2} g(x) = 4$
g(2) is not defined

(C) $\lim_{x\to 0} h(x)$ does not exist
$h(0)$ is not defined

FIGURE 1

$x = c$ is said to be *discontinuous* at $x = c$. Function f (Fig. 1A) is continuous for all x. Function g (Fig. 1B) is discontinuous at $x = 2$, but is continuous over any interval that does not include 2. Function h (Fig. 1C) is discontinuous at $x = 0$, but is continuous over any interval that does not include 0.

Most graphs of natural phenomena are continuous, whereas many graphs in business and economics applications have discontinuities. Figure 2A illustrates temperature variation over a 24-hour period—a continuous phenomenon. Figure 2B illustrates warehouse inventory over a 1-week period—a discontinuous phenomenon.



(A) Temperature for a 24-hour period

(B) Inventory in a warehouse during 1 week

FIGURE 2

Explore & Discuss 1

(A) Write a brief verbal description of the temperature variation illustrated in Figure 2A, including estimates of the high and low temperatures during the period shown and the times at which they occurred.

(B) Write a brief verbal description of the changes in inventory illustrated in Figure 2B, including estimates of the changes in inventory and the times at which those changes occurred.

The preceding discussion leads to the following formal definition of continuity:

DEFINITION Continuity

A function f is **continuous at the point $x = c$ if**

1. $\lim_{x\to c} f(x)$ exists **2.** $f(c)$ exists **3.** $\lim_{x\to c} f(x) = f(c)$

A function is **continuous on the open interval*** **(a, b)** if it is continuous at each point on the interval.

* See Section 1-1 for a review of interval notation.

If one or more of the three conditions in the definition fails, the function is **discontinuous** at $x = c$.

Explore & Discuss **2** Sketch a graph of a function that is discontinuous at a point because it fails to satisfy condition 1 in the definition of continuity. Repeat for conditions 2 and 3.

EXAMPLE 1 **Continuity of a Function Defined by a Graph** Use the definition of continuity to discuss the continuity of the function whose graph is shown in Figure 3.

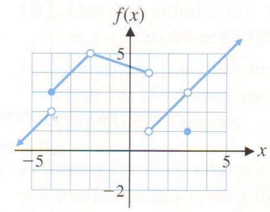


FIGURE 3

SOLUTION We begin by identifying the points of discontinuity. Examining the graph, we see breaks or holes at $x = -4, -2, 1$, and 3. Now we must determine which conditions in the definition of continuity are not satisfied at each of these points. In each case, we find the value of the function and the limit of the function at the point in question.

Discontinuity at $x = -4$:

$$\lim_{x\to -4^-} f(x) = 2$$
$$\lim_{x\to -4^+} f(x) = 3$$
$$\lim_{x\to -4} f(x) \text{ does not exist}$$
$$f(-4) = 3$$

Since the one-sided limits are different, the limit does not exist (Section 10-1).

Thus, f is not continuous at $x = -4$ because condition 1 is not satisfied.

Discontinuity at $x = -2$:

$$\lim_{x\to -2^-} f(x) = 5$$
$$\lim_{x\to -2^+} f(x) = 5$$
$$\lim_{x\to -2} f(x) = 5$$
$$f(-2) \text{ does not exist}$$

The hole at $(-2, 5)$ indicates that 5 is not the value of f at -2. Since there is no solid dot elsewhere on the vertical line $x = -2$, $f(-2)$ is not defined.

Thus, f is not continuous at $x = -2$ because condition 2 is not satisfied.

Discontinuity at $x = 1$:

$$\lim_{x\to 1^-} f(x) = 4$$
$$\lim_{x\to 1^+} f(x) = 1$$
$$\lim_{x\to 1} f(x) \text{ does not exist}$$
$$f(1) \text{ does not exist}$$

This time, f is not continuous at $x = 1$ because neither of conditions 1 and 2 is satisfied.

Discontinuity at $x = 3$:

$$\lim_{x\to 3^-} f(x) = 3$$
$$\lim_{x\to 3^+} f(x) = 3$$
$$\lim_{x\to 3} f(x) = 3$$
$$f(3) = 1$$

The solid dot at $(3, 1)$ indicates that $f(3) = 1$.

Conditions 1 and 2 are satisfied, but f is not continuous at $x = 3$ because condition 3 is not satisfied.

Having identified and discussed all points of discontinuity, we can now conclude that f is continuous except at $x = -4, -2, 1$, and 3.

INSIGHT

Rather than list the points where a function is discontinuous, sometimes it is useful to state the intervals on which the function is continuous. Using the set operation **union,** denoted by $\cup$, we can express the set of points where the function in Example 1 is continuous as follows:

$$(-\infty, -4) \cup (-4, -2) \cup (-2, 1) \cup (1, 3) \cup (3, \infty)$$

MATCHED PROBLEM 1 Use the definition of continuity to discuss the continuity of the function whose graph is shown in Figure 4.

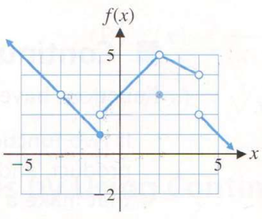


FIGURE 4

For functions defined by equations, it is important to be able to locate points of discontinuity by examining the equation.

EXAMPLE 2 **Continuity of Functions Defined by Equations** Using the definition of continuity, discuss the continuity of each function at the indicated point(s).

(A) $f(x) = x + 2$ at $x = 2$ (B) $g(x) = \dfrac{x^2 - 4}{x - 2}$ at $x = 2$

(C) $h(x) = \dfrac{|x|}{x}$ at $x = 0$ and at $x = 1$

SOLUTION (A) f is continuous at $x = 2$, since

$$\lim_{x \to 2} f(x) = 4 = f(2) \quad \text{See Figure 1A.}$$

(B) g is not continuous at $x = 2$, since $g(2) = 0/0$ is not defined (see Fig. 1B).

(C) h is not continuous at $x = 0$, since $h(0) = |0|/0$ is not defined; also, $\lim_{x \to 0} h(x)$ does not exist.

h is continuous at $x = 1$, since

$$\lim_{x \to 1} \frac{|x|}{x} = 1 = h(1) \quad \text{See Figure 1C.}$$

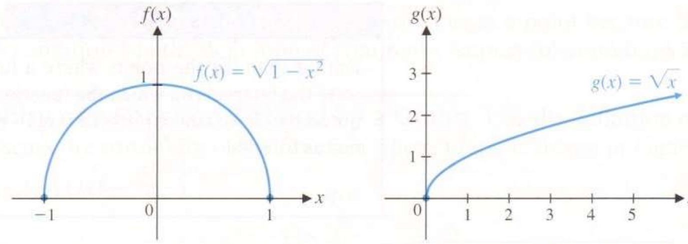
MATCHED PROBLEM 2 Using the definition of continuity, discuss the continuity of each function at the indicated point(s).

(A) $f(x) = x + 1$ at $x = 1$ (B) $g(x) = \dfrac{x^2 - 1}{x - 1}$ at $x = 1$

(C) $h(x) = \dfrac{x - 2}{|x - 2|}$ at $x = 2$ and at $x = 0$

We can also talk about one-sided continuity, just as we talked about one-sided limits. For example, a function is said to be **continuous on the right** at $x = c$ if $\lim_{x \to c^+} f(x) = f(c)$ and **continuous on the left** at $x = c$ if $\lim_{x \to c^-} f(x) = f(c)$. A function is **continuous on the closed interval [a, b]** if it is continuous on the open interval (a, b) and is continuous both on the right at a and on the left at b.

Figure 5A illustrates a function that is continuous on the closed interval $[-1, 1]$. Figure 5B illustrates a function that is continuous on the half-closed interval $[0, \infty)$.



(A) f is continuous on the closed interval $[-1, 1]$

(B) g is continuous on the half-closed interval $[0, \infty)$

FIGURE 5 Continuity on closed and half-closed intervals

Continuity Properties

Functions have some useful **general continuity properties:**

If two functions are continuous on the same interval, then their sum, difference, product, and quotient are continuous on the same interval, except for values of x that make a denominator 0.

These properties, along with Theorem 1 below, enable us to determine intervals of continuity for some important classes of functions without having to look at their graphs or use the three conditions in the definition.

THEOREM 1 CONTINUITY PROPERTIES OF SOME SPECIFIC FUNCTIONS

(A) A constant function $f(x) = k$, where k is a constant, is continuous for all x.

$f(x) = 7$ is continuous for all x.

(B) For n a positive integer, $f(x) = x^n$ is continuous for all x.

$f(x) = x^5$ is continuous for all x.

(C) A polynomial function is continuous for all x.

$2x^3 - 3x^2 + x - 5$ is continuous for all x.

(D) A rational function is continuous for all x except those values that make a denominator 0.

$\dfrac{x^2 + 1}{x - 1}$ is continuous for all x except $x = 1$, a value that makes the denominator 0.

(E) For n an odd positive integer greater than 1, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous.

$\sqrt[3]{x^2}$ is continuous for all x.

(F) For n an even positive integer, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous and nonnegative.

$\sqrt[4]{x}$ is continuous on the interval $[0, \infty)$.

Parts C and D of Theorem 1 are the same as Theorem 3 in Section 10-1. They are repeated here to emphasize their importance.

EXAMPLE 3 **Using Continuity Properties** Using Theorem 1 and the general properties of continuity, determine where each function is continuous.

(A) $f(x) = x^2 - 2x + 1$

(B) $f(x) = \dfrac{x}{(x + 2)(x - 3)}$

(C) $f(x) = \sqrt[3]{x^2 - 4}$

(D) $f(x) = \sqrt{x - 2}$

SOLUTION (A) Since f is a polynomial function, f is continuous for all x.

(B) Since f is a rational function, f is continuous for all x except -2 and 3 (values that make the denominator 0).

(C) The polynomial function $x^2 - 4$ is continuous for all x. Since $n = 3$ is odd, f is continuous for all x.

(D) The polynomial function $x - 2$ is continuous for all x and nonnegative for $x \geq 2$. Since $n = 2$ is even, f is continuous for $x \geq 2$, or on the interval $[2, \infty)$.

MATCHED PROBLEM 3 Using Theorem 1 and the general properties of continuity, determine where each function is continuous.

(A) $f(x) = x^4 + 2x^2 + 1$ (B) $f(x) = \dfrac{x^2}{(x + 1)(x - 4)}$

(C) $f(x) = \sqrt{x - 4}$ (D) $f(x) = \sqrt[3]{x^3 + 1}$

Solving Inequalities by Using Continuity Properties

One of the basic tools for analyzing graphs in calculus is a special line graph called a *sign chart.* We will make extensive use of this type of chart in later sections. In the discussion that follows, we use continuity properties to develop a simple and efficient procedure for constructing sign charts.

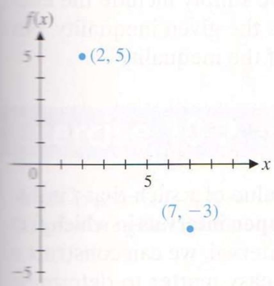


FIGURE 6

Suppose that a function f is continuous over the interval $(1, 8)$ and $f(x) \neq 0$ for any x in $(1, 8)$. Suppose also that $f(2) = 5$, a positive number. Is it possible for $f(x)$ to be negative for any x in the interval $(1, 8)$? The answer is "no." If $f(7)$ were -3, for example, as shown in Figure 6, how would it be possible to join the points $(2, 5)$ and $(7, -3)$ with the graph of a continuous function without crossing the x axis between 1 and 8 at least once? [Crossing the x axis would violate our assumption that $f(x) \neq 0$ for any x in $(1, 8)$.] Thus, we conclude that $f(x)$ must be positive for all x in $(1, 8)$. If $f(2)$ were negative, then, using the same type of reasoning, $f(x)$ would have to be negative over the entire interval $(1, 8)$.

In general, **if f is continuous and $f(x) \neq 0$ on the interval (a, b), then $f(x)$ cannot change sign on (a, b).** This is the essence of Theorem 2.

THEOREM 2 SIGN PROPERTIES ON AN INTERVAL (a, b)

If f is continuous on (a, b) and $f(x) \neq 0$ for all x in (a, b), then either $f(x) > 0$ for all x in (a, b) or $f(x) < 0$ for all x in (a, b).

Theorem 2 provides the basis for an effective method of solving many types of inequalities. Example 4 illustrates the process.

EXAMPLE 4 **Solving an Inequality** Solve $\dfrac{x + 1}{x - 2} > 0$.

SOLUTION We start by using the left side of the inequality to form the function f:

$$f(x) = \frac{x + 1}{x - 2}$$

The rational function f is discontinuous at $x = 2$, and $f(x) = 0$ for $x = -1$ (a fraction is 0 when the numerator is 0 and the denominator is not 0). We plot $x = 2$ and $x = -1$, which we call *partition numbers,* on a real-number line (Fig. 7). (Note that the dot at 2 is open, because the function is not defined at $x = 2$.) The partition numbers 2 and -1 determine three open intervals: $(-\infty, -1)$, $(-1, 2)$, and $(2, \infty)$. The function f is continuous and nonzero on each of these intervals. From Theorem 2, we know that $f(x)$ does not change sign on any of these intervals. Thus, we can find

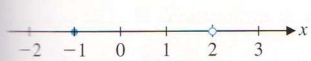
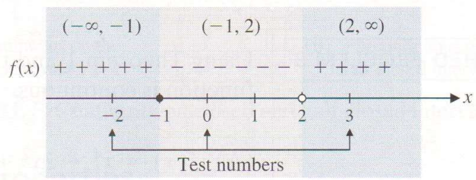


FIGURE 7

Test Numbers	
x	$f(x)$
-2	$\frac{1}{4}$ (+)
0	$-\frac{1}{2}$ (−)
3	4 (+)

the sign of $f(x)$ on each of the intervals by selecting a **test number** in each interval and evaluating $f(x)$ at that number. Since any number in each subinterval will do, we choose test numbers that are easy to evaluate: -2, 0, and 3. The table in the margin shows the results.

The sign of $f(x)$ at each test number is the same as the sign of $f(x)$ over the interval containing that test number. Using this information, we construct a **sign chart** for $f(x)$:



From the sign chart, we can easily write the solution of the given nonlinear inequality:

$$f(x) > 0 \quad \text{for} \quad \begin{array}{ll} x < -1 \text{ or } x > 2 & \text{Inequality notation} \\ (-\infty, -1) \cup (2, \infty) & \text{Interval notation} \end{array}$$

Most of the inequalities we encounter will involve strict inequalities ($>$ or $<$). If it is necessary to solve inequalities of the form $\geq$ or $\leq$, we simply include the endpoint x of any interval if f is defined at x and $f(x)$ satisfies the given inequality. For example, from the sign chart in Example 4, the solution of the inequality

$$\frac{x+1}{x-2} \geq 0 \quad \text{is} \quad \begin{array}{ll} x \leq -1 \text{ or } x > 2 & \text{Inequality notation} \\ (-\infty, -1] \cup (2, \infty) & \text{Interval notation} \end{array}$$

In general, given a function f, a **partition number** is a value of x such that f is discontinuous at x or $f(x) = 0$. **Partition numbers determine open intervals in which $f(x)$ does not change sign.** By using a test number from each interval, we can construct a sign chart for $f(x)$ on the real-number line. It is then an easy matter to determine where $f(x) < 0$ or $f(x) > 0$; that is, it is easy to solve the inequality $f(x) < 0$ or $f(x) > 0$.

We summarize the procedure for constructing sign charts in the following box:

PROCEDURE **Constructing Sign Charts**

Given a function f,

Step 1. Find all partition numbers. That is,

(A) Find all numbers such that f is discontinuous. (Rational functions are discontinuous for values of x that make a denominator 0.)

(B) Find all numbers such that $f(x) = 0$. (For a rational function, this occurs where the numerator is 0 and the denominator is not 0.)

Step 2. Plot the numbers found in step 1 on a real-number line, dividing the number line into intervals.

Step 3. Select a test number in each open interval determined in step 2, and evaluate $f(x)$ at each test number to determine whether $f(x)$ is positive (+) or negative (−) in each interval.

Step 4. Construct a sign chart, using the real-number line in step 2. This will show the sign of $f(x)$ on each open interval.

Note: From the sign chart, it is easy to find the solution of the inequality $f(x) < 0$ or $f(x) > 0$.

MATCHED PROBLEM 4 Solve $\dfrac{x^2 - 1}{x - 3} < 0$.

Answers to Matched Problems

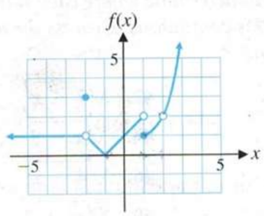
1. f is not continuous at $x = -3, -1, 2$, and 4.
   $x = -3$: $\lim_{x\to-3} f(x) = 3$, but $f(-3)$ does not exist
   $x = -1$: $f(-1) = 1$, but $\lim_{x\to-1} f(x)$ does not exist
   $x = 2$: $\lim_{x\to2} f(x) = 5$, but $f(2) = 3$
   $x = 4$: $\lim_{x\to4} f(x)$ does not exist, and $f(4)$ does not exist
2. (A) f is continuous at $x = 1$, since $\lim_{x\to1} f(x) = 2 = f(1)$.
   (B) g is not continuous at $x = 1$, since $g(1)$ is not defined.
   (C) h is not continuous at $x = 2$ for two reasons: $h(2)$ does not exist and $\lim_{x\to2} h(x)$ does not exist.
   h is continuous at $x = 0$, since $\lim_{x\to0} h(x) = -1 = h(0)$.
3. (A) Since f is a polynomial function, f is continuous for all x.
   (B) Since f is a rational function, f is continuous for all x except -1 and 4 (values that make the denominator 0).
   (C) The polynomial function $x - 4$ is continuous for all x and nonnegative for $x \ge 4$. Since $n = 2$ is even, f is continuous for $x \ge 4$, or on the interval $[4, \infty)$.
   (D) The polynomial function $x^3 + 1$ is continuous for all x. Since $n = 3$ is odd, f is continuous for all x.
4. $-\infty < x < -1$ or $1 < x < 3$; $(-\infty, -1) \cup (1, 3)$

Exercise 10-2

In Problems 1–6, sketch a possible graph of a function that satisfies the given conditions at $x = 1$, and discuss the continuity of f at $x = 1$.

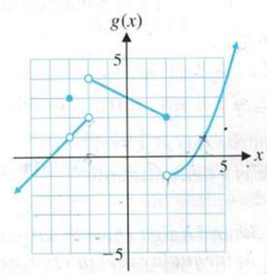
1. $f(1) = 2$ and $\lim_{x\to1} f(x) = 2$
2. $f(1) = -2$ and $\lim_{x\to1} f(x) = 2$
3. $f(1) = 2$ and $\lim_{x\to1} f(x) = -2$
4. $f(1) = -2$ and $\lim_{x\to1} f(x) = -2$
5. $f(1) = -2$, $\lim_{x\to1^-} f(x) = 2$, and $\lim_{x\to1^+} f(x) = -2$
6. $f(1) = 2$, $\lim_{x\to1^-} f(x) = 2$, and $\lim_{x\to1^+} f(x) = -2$

Problems 7–10 refer to the function f shown in the figure. Use the graph to estimate the indicated quantities to the nearest integer.



7. (A) $\lim_{x\to1^-} f(x)$ (B) $\lim_{x\to1^+} f(x)$
   (C) $\lim_{x\to1} f(x)$ (D) $f(1)$
   (E) Is f continuous at $x = 1$? Explain.
8. (A) $\lim_{x\to2^-} f(x)$ (B) $\lim_{x\to2^+} f(x)$
   (C) $\lim_{x\to2} f(x)$ (D) $f(2)$
   (E) Is f continuous at $x = 2$? Explain.
9. (A) $\lim_{x\to-2^-} f(x)$ (B) $\lim_{x\to-2^+} f(x)$
   (C) $\lim_{x\to-2} f(x)$ (D) $f(-2)$
   (E) Is f continuous at $x = -2$? Explain.
10. (A) $\lim_{x\to-1^-} f(x)$ (B) $\lim_{x\to-1^+} f(x)$
    (C) $\lim_{x\to-1} f(x)$ (D) $f(-1)$
    (E) Is f continuous at $x = -1$? Explain.

Problems 11–14 refer to the function g shown in the figure. Use the graph to estimate the indicated quantities to the nearest integer.



11. (A) $\lim_{x\to-3^-} g(x)$ (B) $\lim_{x\to-3^+} g(x)$
    (C) $\lim_{x\to-3} g(x)$ (D) $g(-3)$
    (E) Is g continuous at $x = -3$? Explain.
12. (A) $\lim_{x\to-2^-} g(x)$ (B) $\lim_{x\to-2^+} g(x)$
    (C) $\lim_{x\to-2} g(x)$ (D) $g(-2)$
    (E) Is g continuous at $x = -2$? Explain.

13. (A) $\lim_{x \to 2^-} g(x)$ (B) $\lim_{x \to 2^+} g(x)$
(C) $\lim_{x \to 2} g(x)$ (D) $g(2)$
(E) Is g continuous at $x = 2$? Explain.

14. (A) $\lim_{x \to 4^-} g(x)$ (B) $\lim_{x \to 4^+} g(x)$
(C) $\lim_{x \to 4} g(x)$ (D) $g(4)$
(E) Is g continuous at $x = 4$? Explain.

Use Theorem 1 to determine where each function in Problems 15–24 is continuous.

15. $f(x) = 3x - 4$

16. $h(x) = 4 - 2x$

17. $g(x) = \dfrac{3x}{x + 2}$

18. $k(x) = \dfrac{2x}{x - 4}$

19. $m(x) = \dfrac{x + 1}{(x - 1)(x + 4)}$

20. $n(x) = \dfrac{x - 2}{(x - 3)(x + 1)}$

21. $F(x) = \dfrac{2x}{x^2 + 9}$

22. $G(x) = \dfrac{1 - x^2}{x^2 + 1}$

23. $M(x) = \dfrac{x - 1}{4x^2 - 9}$

24. $N(x) = \dfrac{x^2 + 4}{4 - 25x^2}$

B

25. Given the function

$$f(x) = \begin{cases} 2 & \text{if } x \text{ is an integer} \\ 1 & \text{if } x \text{ is not an integer} \end{cases}$$

(A) Graph f.
(B) $\lim_{x \to 2} f(x) = ?$
(C) $f(2) = ?$
(D) Is f continuous at $x = 2$?
(E) Where is f discontinuous?

26. Given the function

$$g(x) = \begin{cases} -1 & \text{if } x \text{ is an even integer} \\ 1 & \text{if } x \text{ is not an even integer} \end{cases}$$

(A) Graph g.
(B) $\lim_{x \to 1} g(x) = ?$
(C) $g(1) = ?$
(D) Is g continuous at $x = 1$?
(E) Where is g discontinuous?

In Problems 27–34, use a sign chart to solve each inequality. Express answers in inequality and interval notation.

27. $x^2 - x - 12 < 0$

28. $x^2 - 2x - 8 < 0$

29. $x^2 + 21 > 10x$

30. $x^2 + 7x > -10$

31. $x^3 < 4x$

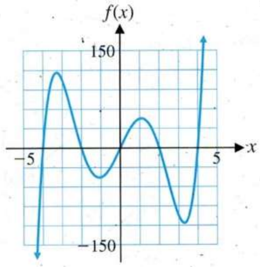
32. $x^4 - 9x^2 > 0$

33. $\dfrac{x^2 + 5x}{x - 3} > 0$

34. $\dfrac{x - 4}{x^2 + 2x} < 0$

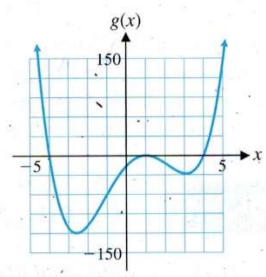
35. Use the graph of f to determine where
(A) $f(x) > 0$ (B) $f(x) < 0$

Express answers in interval notation.



36. Use the graph of g to determine where
(A) $g(x) > 0$ (B) $g(x) < 0$

Express answers in interval notation.



In Problems 37–40, use a graphing calculator to approximate the partition numbers of each function f(x) to four decimal places. Then solve the following inequalities:

(A) $f(x) > 0$ (B) $f(x) < 0$

Express answers in interval notation.

37. $f(x) = x^4 - 6x^2 + 3x + 5$

38. $f(x) = x^4 - 4x^2 - 2x + 2$

39. $f(x) = \dfrac{3 + 6x - x^3}{x^2 - 1}$

40. $f(x) = \dfrac{x^3 - 5x + 1}{x^2 - 1}$

Use Theorem 1 to determine where each function in Problems 41–48 is continuous. Express the answer in interval notation.

41. $\sqrt{x - 6}$

42. $\sqrt{7 - x}$

43. $\sqrt[3]{5 - x}$

44. $\sqrt[3]{x - 8}$

45. $\sqrt{x^2 - 9}$

46. $\sqrt{4 - x^2}$

47. $\sqrt{x^2 + 1}$

48. $\sqrt[3]{x^2 + 2}$

In Problems 49–54, graph f, locate all points of discontinuity and discuss the behavior of f at these points.

49. $f(x) = \begin{cases} 1 + x & \text{if } x < 1 \\ 5 - x & \text{if } x \geq 1 \end{cases}$

50. $f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x & \text{if } x > 1 \end{cases}$

51. $f(x) = \begin{cases} 1 + x & \text{if } x \le 2 \\ 5 - x & \text{if } x > 2 \end{cases}$

52. $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ 2x & \text{if } x > 2 \end{cases}$

53. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$

54. $f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 + x & \text{if } x > 0 \end{cases}$

In Problems 55–58, use a graphing calculator to locate all points of discontinuity of f, and discuss the behavior of f at these points. [Hint: Select Xmin and Xmax so that the suspected point of discontinuity is the midpoint of the graphing interval (Xmin, Xmax).]

55. $f(x) = x + \dfrac{|2x - 4|}{x - 2}$

56. $f(x) = x + \dfrac{|3x + 9|}{x + 3}$

57. $f(x) = \dfrac{x^2 - 1}{|x| - 1}$

58. $f(x) = \dfrac{x^3 - 8}{|x| - 2}$

C

59. Use the graph of the function g to answer the following questions:
    (A) Is g continuous on the open interval $(-1, 2)$?
    (B) Is g continuous from the right at $x = -1$? That is, does $\lim_{x \to -1^+} g(x) = g(-1)$?
    (C) Is g continuous from the left at $x = 2$? That is, does $\lim_{x \to 2^-} g(x) = g(2)$?
    (D) Is g continuous on the closed interval $[-1, 2]$?

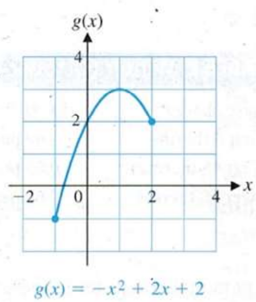


Figure for 59

60. Use the graph of the function f to answer the following questions:
    (A) Is f continuous on the open interval $(0, 3)$?
    (B) Is f continuous from the right at $x = 0$? That is, does $\lim_{x \to 0^+} f(x) = f(0)$?
    (C) Is f continuous from the left at $x = 3$? That is, does $\lim_{x \to 3^-} f(x) = f(3)$?
    (D) Is f continuous on the closed interval $[0, 3]$?

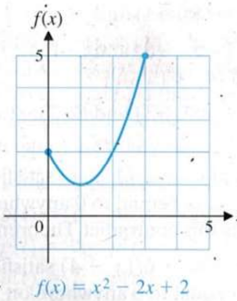


Figure for 60

Problems 61 and 62 refer to the ***greatest integer function,*** *which is denoted by* $[\![x]\!]$ *and is defined as*

$$[\![x]\!] = \text{greatest integer} \le x$$

For example,

$$[\![-3.6]\!] = \text{greatest integer} \le -3.6 = -4$$
$$[\![2]\!] = \text{greatest integer} \le 2 = 2$$
$$[\![2.5]\!] = \text{greatest integer} \le 2.5 = 2$$

The graph of $f(x) = [\![x]\!]$ *is shown. There, we can see that*

$$\begin{aligned} [\![x]\!] &= -2 \quad \text{for} \quad -2 \le x < -1 \\ [\![x]\!] &= -1 \quad \text{for} \quad -1 \le x < 0 \\ [\![x]\!] &= 0 \quad \text{for} \quad 0 \le x < 1 \\ [\![x]\!] &= 1 \quad \text{for} \quad 1 \le x < 2 \\ [\![x]\!] &= 2 \quad \text{for} \quad 2 \le x < 3 \end{aligned}$$

and so on.

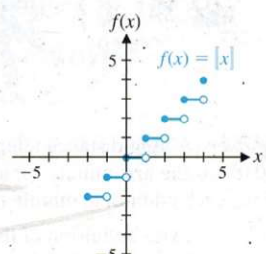


Figure for 61 and 62

61. (A) Is f continuous from the right at $x = 0$?
    (B) Is f continuous from the left at $x = 0$?
    (C) Is f continuous on the open interval $(0, 1)$?
    (D) Is f continuous on the closed interval $[0, 1]$?
    (E) Is f continuous on the half-closed interval $[0, 1)$?

62. (A) Is f continuous from the right at $x = 2$?
    (B) Is f continuous from the left at $x = 2$?
    (C) Is f continuous on the open interval $(1, 2)$?
    (D) Is f continuous on the closed interval $[1, 2]$?
    (E) Is f continuous on the half-closed interval $[1, 2)$?

In Problems 63–66, sketch a possible graph of a function f that is continuous for all real numbers and satisfies the given conditions. Find the x intercepts of f.

63. $f(x) < 0$ on $(-\infty, -5)$ and $(2, \infty)$; $f(x) > 0$ on $(-5, 2)$